1 Theoretical questions

1. Fundamental system properties

   (a) Define the notion of a linear system.
   The $S : \mathcal{U} \mapsto \mathcal{Y}$ system is linear, if

   $$S[\alpha u_1 + \beta u_2] = \alpha S[u_1] + \beta S[u_2]$$

   holds $\forall u_1, u_2 \in \mathcal{U}, \alpha, \beta \in \mathbb{R}$.

   (b) Is the following system model time invariant? Why?

   $$\begin{aligned} \dot{x}(t) &= -x^3(t) + \cos(t)u(t) \\ y(t) &= 0.5x(t). \end{aligned}$$

   The system is not time invariant, because the coefficient of the input $u(t)$ is time dependent.

2. How can we compute the impulse response function of a system given in the form of a state space model with matrices A, B and C? What is the explicit form of $h(t)$? We assume that $D = 0$.
   We can compute $h(t)$ from $H(s)$ using inverse Laplace transformation, i.e.

   $$h(t) = \mathcal{L}^{-1}\big\{H(s)\big\} = \mathcal{L}^{-1}\big\{C(sI - A)^{-1}B\big\}.$$

   The explicit form of $h(t)$ is

   $$h(t) = Ce^{At}B.$$

3. Consider the following state-space model

   $$\begin{aligned} \dot{x} &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x \end{aligned}$$

   where $\forall \lambda_i \in \mathbb{R}$.

   (a) Give a base of the controllable subspace of the system.
   The controllable subspace of the system is the image space (column space) of the controllability matrix $\mathcal{C}_n$.

   $$C_n = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 0 & 0 & 0 \end{bmatrix}$$

   From this

   $$\operatorname{Im} \mathcal{C}_n = \begin{cases} \operatorname{span}\left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}} \right) & \text{if } \lambda_1 \neq \lambda_2 \\ \operatorname{span}\left(\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathrm{T}} \right) & \text{if } \lambda_1 = \lambda_2 \end{cases}.$$

   (b) How can we select $\lambda_1, \lambda_2, \lambda_3$ such that the dimension of the controllable subspace is 1?
   Based on the previous exercise if $\lambda_1 = \lambda_2$ and $\lambda_3 \in \mathbb{R}$, then the controllable subspace will be of dimension 1.

4. Give the controller form realization of the following transfer function model

   $$H(s) = \frac{s^2 - 1}{(s+2)(s-2)(s+3)}.$$

Is the computed realization observable? Why?
For the controller form realization we need to know the coefficients of the polynomials $a(s)$ and $b(s)$, where $H(s) = \frac{b(s)}{a(s)}$.

$$H(s) = \frac{s^2-1}{(s+2)(s-2)(s+3)} = \frac{s^2-1}{s^3+3s^2-4s-12}$$

From this the realization is

$$\dot{x} = \begin{bmatrix} -3 & 4 & 12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} x.$$

The computed realization is jointly controllable and observable, since $H(s)$ is irreducible (i.e. it is a minimal realization).

2 Computational questions

1. A linear state space model is given with the following matrices

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

(a) Can we determine the value of the state vector from a finite measurement of the inputs and outputs?
The question is the definition of observability, i.e. we need to check the rank of the observability matrix.

$$\operatorname{rank} \mathcal{O}_n = \operatorname{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = 2$$

since $\det \mathcal{O}_n \neq 0$.

(b) Determine the output of the system if the input is the Dirac-delta function.
If the input is $\delta(t)$, then the output is $h(t)$, i.e. we need to determine the impulse response function. We will do two ways, first with the equation $h(t) = \mathcal{L}^{-1}\{C(sI-A)^{-1}B\}$, then with the equation $h(t) = Ce^{At}B$.

i.

$$H(s) = C(sI-A)^{-1}B = C \begin{bmatrix} s+2 & 0 \\ -1 & s+1 \end{bmatrix}^{-1} B = C \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{(s+1)} \end{bmatrix} B =$$
$$= \begin{bmatrix} \frac{1}{s+2} + \frac{2}{(s+1)(s+2)} & \frac{2}{s+1} \end{bmatrix} B = \frac{1}{s+2} - \frac{2}{s+1} + \frac{2}{(s+1)(s+2)} =$$
$$= \frac{1}{s+2} - \frac{2}{s+1} + \frac{2}{s+1} - \frac{2}{s+2} = -\frac{1}{s+2}$$
$$h(t) = \mathcal{L}^{-1}\{H(s)\} = -e^{-2t}$$

ii. For the second method first we will need to diagonalize the matrix A, i.e. we need to find the matrices D, S such that $D = S^{-1}AS$. For this we need the eigenvalues and eigenvectors of A. The matrix D will be a diagonal matrix containing the eigenvalues and the matrix S will contain the corresponding eigenvalues as columns.
The eigenvalues are obviously $\lambda_1 = -1$ and $\lambda_2 = -2$, since A is a triangular matrix. The eigenvectors are

$$(\lambda_1 I - A)v = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies s_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$(\lambda_2 I - A)v = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -v_1 - v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies s_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

From this with $S = \begin{bmatrix} s_1 & s_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = S^{-1}AS$$

i.e. $A = SDS^{-1}$. We will use this to compute the exponential matrix $e^{At} = Se^{Dt}S^{-1}$.

$$e^{Dt} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$
$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 0 & e^{-2t} \\ e^{-t} & -e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

From this

$$h(t) = Ce^{At}B = -e^{-2t}.$$

(c) Give a particular value of the state vector x that cannot be reached by any input from the origin. We need to give a particular point from the uncontrollable subspace, which is $\operatorname{Ker} \mathcal{C}_n^{\mathrm{T}}$.

$$\mathcal{C}_n^{\mathrm{T}} = \begin{bmatrix} B & AB \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$
$$\mathcal{C}_n^{\mathrm{T}} x = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 = x_2$$

i.e. the uncontrollable subspace is the set of vectors $\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $\alpha \in \mathbb{R}$. Thus an unreachable state vector is e.g. $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. We consider a continuous-time LTI system given by its transfer function

$$H(s) = \frac{s^2 - 1}{(s^2 + 5s + 6)(2s - 2)}.$$

(a) Give a jointly controllable and observable state-space realization for this system. Notice that $H(s)$ is reducible

$$H(s) = \frac{s^2 - 1}{(s^2 + 5s + 6)(2s - 2)} = \frac{s + 1}{2(s^2 + 5s + 6)} = \frac{\frac{1}{2}s + \frac{1}{2}}{s^2 + 5s + 6}.$$

In this form $H(s)$ is irreducible, i.e. every two dimensional realization will be minimal, i.e. jointly controllable and observable. E.g. the observer form realization is

$$\dot{x} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

(b) Determine whether the system is BIBO stable or not. The eigenvalues of the matrix A are -2 and -3 (these are the poles of the transfer function), i.e. the system is asymptotically stable, which implies BIBO stability.

3. Compute the step response of the following system given by its transfer function

$$H(s) = \frac{s}{s^2 + 3s + 2}.$$

We can do this using Laplace transform

$$Y(s) = H(s)U(s) = \frac{s}{s^2 + 3s + 2}\frac{1}{s} = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-t} - e^{-2t}.$$

4. Given the following nonlinear state-space model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - \sin 2x_1$$

compute a suitable $a \in \mathbb{R}$, such that the following function is a Lyapunov function

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) + a \sin^2 x_1.$$

(a) We need that $V \in \mathcal{C}^1$, i.e. it is continuously differentiable. This is true.

(b) We need that $\forall x \neq 0 \in \mathbb{R}^2 : V(x) > 0$. This is true if $a > 0$.

(c) We need that $\forall x \neq 0 \in \mathbb{R}^2 : \frac{\mathrm{d}}{\mathrm{d}x} V(x) \leq 0$.

$$\frac{\mathrm{d}}{\mathrm{d}x} V(x) = \langle \nabla V, \dot{x} \rangle = \begin{bmatrix} x_1 + 2a \sin x_1 \cos x_1 \\ x_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x_2 \\ -x_1 - x_2 - \sin 2x_1 \end{bmatrix} =$$
$$= x_1 x_2 + 2a x_2 \sin x_1 \cos x_1 - x_1 x_2 - x_2^2 - x_2 \sin 2x_1 = (a-1) x_2 \sin 2x_1 - x_2^2$$

For this to be nonnegative we need $a = 1$.

Thus $V(x)$ is a Lyapunov function if $a = 1$.